MATH 1720.620
CALCULUS II
SPRING 2019

Scientia Imperii Decus et Tutamen ¹

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¹Taken from the coat of arms of Imperial College London.
SPRING 2019 COURSE: MATH 1720.620, CALCULUS II.

PREREQUISITES: MATH 1710.

CLASS MEETS: Monday, Wednesday, Friday 10:00 a.m. - 10:50 a.m., CURY 211.

FINAL EXAM DATE AND TIME: The final is scheduled for Saturday, May 4, 2019 in CURY 211, 8:00 a.m. - 10:00 a.m.


INSTRUCTOR: Robert R. Kallman, 315 GAB [office], 940-565-3329 [office telephone], 940-565-4805 [fax], robert.kallman@unt.edu [e-mail]

OFFICE HOURS: Monday, Wednesday, Friday, 8:00 a.m. - 9:30 a.m. & 11:00 a.m. - 12:30 pm

ATTENDANCE POLICY: Mandatory. Specifically for TAMS students: if you are absent for any reason, you are required to file an absence report with the appropriate official in the TAMS Academic Office.

ELECTRONIC DEVICES: No electronic devices of any sort are to be on let alone used during the class. Repeated flouting of this will result in a grade penalty.

HOMEWORK: Homework will be assigned and some designated subset of it will be graded. The designated homework assigned on Monday, Wednesday and Friday of one week will be due at the beginning of class on the Wednesday of the following week. Late homeworks will not be accepted under any circumstances. This applies even if the student is ill - give your homework to a classmate to hand in on time or convert your homework into a *.pdf file and e-mail to me with a time stamp earlier than the beginning of class. Each homework problem will receive a grade of 0, 1, or 2 points. Failure to turn in a homework set on time will result in a grade of -1 for that homework set.

ACADEMIC INTEGRITY: There is no reason why TAMS students should not demonstrate complete academic integrity at all times, particularly on tests. Any transgression of this will result in a grade of zero on the test and a grade of F for the course. Consistent with this policy the instructor will retain xerox copies of a random sampling of all tests.
**GRADING POLICY:** Grades will be based on the total number of points accrued from the assigned graded homeworks, from two in class one hour examinations (5 problems plus 1 bonus question), given circa in late February and late April, and from an in class 120 minute final (8 problems plus 1 bonus question). The number of points per test and final problem will normally be 10. There is no excuse for missing a test and no makeup tests will be given under any circumstances. A student missing a test will receive a grade of -1 on that test. If a student is unavoidably absent from a test and makes arrangements with the instructor well before the test date, then the grade assigned to the missing test will be prorated by the student’s performance on the final examination minus 10 points. It is difficult a priori to determine the precise break points for the final grades. However, the golden rule in determining the final assigned grade is that if the number of points earned by person A is $\geq$ to the number of points earned by person B, then person A has a grade which is $\geq$ to the grade of person B. The only possible exception is that you must take the final examination and receive a passing grade on the final in order to get a grade greater than F.

Just as in last semester, students can obtain a real time good first approximation to their progress/status in this class by doing the following calculation. Let $S = 0.4(A/B) + 0.6(C/D)$, where $A =$ number of homework points earned to date, $B =$ maximum number of homework points possible to date, $C =$ number of test and final points earned to date and $D =$ maximum number of test points possible to date (not counting bonus points). Note that $0.0 \leq S \leq 1.0$. Then the grade in this class at this instant is: A if $S \geq 0.9$; B if $0.8 \leq S < 0.9$; C if $0.7 \leq S < 0.8$; D if $0.6 \leq S < 0.7$; F if $S < 0.6$.

**TOPICS:** The topics to be covered can be found in most of Chapter 6.1 - 6.6, 6.8, 7.1 - 7.5, 7.8, 8.1, 8.2, 9.3, 9.5 and an independent study of Taylor’s formula with remainder. It is a rather ambitious goal to cover these topics in some depth. This will require considerable work on the part of the students and the instructor. Some supplementary notes will be handed out.

**APPROXIMATE ITINERARY:** The following is a first attempt, very rough approximation to what our schedule will be. This will perhaps be dynamically reconfigured as the semester progresses since it is of course impossible to make such a schedule with hard-and-fast rules.

Test #1, Friday, February 22, 2019, covering 6.1 - 6.6, 6.8, 7.1 and 7.2.

Test #2, Friday, April 12, 2019, covering 7.3 - 7.5, 7.8, 8.1, 8.2, 9.3, 9.5, Selected portions of Chapter 11, especially an independent study of Taylor’s Formula with the integral form of the remainder.
Final, Saturday, May 4, 2019, cumulative.

**ASK QUESTIONS** in class so that we may all benefit. If you need help, it is your responsibility to seek me out. See me during my office hours. Empirical evidence suggests that there is a strong correlation between the amount of work done by the student and his/her final grade.

**STUDENTS WITH DISABILITIES:** It is the responsibility of students with certified disabilities to provide the instructor with appropriate documentation from the Dean of Students Office.

**STUDENT BEHAVIOR IN THE CLASSROOM:** The Powers That Be have strongly suggested that students be given the following statement:

Student behavior that interferes with an instructor’s ability to conduct a class or other students’ opportunity to learn is unacceptable and disruptive and will not be tolerated in any instructional forum at UNT. Students engaging in unacceptable behavior will be directed to leave the classroom and the instructor may refer the student to the Center for Student Rights and Responsibilities to consider whether the student’s conduct violated the Code of Student Conduct. The university’s expectations for student conduct apply to all instructional forums, including university and electronic classroom, labs, discussion groups, field trips, etc. The Code of Student Conduct can be found at www.unt.edu/csrr

In other words, cause trouble in the classroom and you will probably be cast into the Darkness and sent to the KGB.
How to Study for This Class

- Attend every class.

- Pay attention in class, take careful notes, and ask questions if needed.

- The evening of every class go over your classroom notes, list topics on which you have questions or need clarification, read the relevant section of the textbook, do the assigned homework to be graded, look over the additional homeworks to verify that you understand how to do them and make note of those additional homeworks that you do not understand to ask about them during the next class. It is important that you put a great deal of effort into the homework, both those to be turned in and those that are less formally assigned. One becomes adroit at any human activity - e.g., hitting a fast ball, throwing a slider, making foul shots or jump shots, or driving off a tee - only with a great deal of practice. The same applies to calculus.

- Do not waste your time memorizing endless lists of derivatives and antiderivatives. This in fact is counterproductive. Instead, know a few basic computational techniques (e.g., product rule for differentiation, chain rule for differentiation, sin' = cos, etc.), and try to understand the big picture and concepts involved in problem solving. All of the problems encountered in this class should be first approached by asking oneself what is a reasonable way to proceed. Then given the proper path or direction, you can then solve the problems by small, logical steps that inevitably lead one to the final solution.
Recall essential facts from 1710 and before: \( f' \) or \( Df \) given by 
\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \]

\((f + g)' = f' + g', (af)' = af', (fg)' = f'g + fg' \) (Product Rule), 
\( (f/g)' = (f'g - fg')/g^2 \) (Quotient Rule), 
\( h = f \circ g \implies h' = (f' \circ g)g' \) (Chain Rule), 
\( (x^n)' = nx^{n-1} \), derivatives of polynomials and rational functions, 
\( (f^n)' = nf^{n-1}f' \), addition formulas for \( \sin \) and \( \cos \), 
double angle formulas, and half angle formulas, 
\( \cos^2(x) + \sin^2(x) = 1 \), \( \sin' = \cos \), \( \cos' = -\sin \), 
\( \tan' = \sec^2 \), \( \sec'(x) = \sec(x) \tan(x) \), 
Fundamental Theorems of Calculus, continuous functions assume maximum and minimum on closed bounded intervals, 
local maxima and minima on open intervals, Rolle’s Theorem, 
Mean Value Theorem, \( f' = 0 \implies f \) is a constant, \( f' > 0 \implies f \uparrow \), \( f' < 0 \implies f \downarrow \), 
\( F_1' = F_2' \implies F_1 = F_2 + C \), one-to-one onto functions and their inverses and derivatives. 
Volumes of solids of rotation and arc length formula. 
The surface area of a cone and the surface area generated by a surface of revolution. 
Volume and surface area of a sphere and spherical caps and a circumscribing cylinder, recalling profound work of Archimedes. 
L'Hôpital’s Rule.
Theorem 1 (Darboux).
Let \( a, b \in \mathbb{R} \) and let \( f : [a,b] \to \mathbb{R} \) be differentiable. If \( \lambda \in \mathbb{R} \) is between \((Df)(a)\) and \((Df)(b)\), then there exists \( c \in (a,b) \) such that \((Df)(c) = \lambda\), i.e., even though \(Df\) need not be continuous, it does have the intermediate value property.

Proof:
If \((Df)(a) = (Df)(b)\), there is nothing to do. Otherwise we may suppose that \((Df)(a) < (Df)(b)\), for if not, replace \(f\) with \(-f\) and \(\lambda\) with \(-\lambda\). Let \(g(t) = f(t) - \lambda t\). Then \((Dg)(a) < 0\) and \((Dg)(b) > 0\), so that \(g(t_1) < g(a)\) for some \(t_1 \in (a,b)\) and \(g(t_2) < g(b)\) for some \(t_2 \in (a,b)\). Hence, \(g\) assumes its minimum on \([a,b]\) at some point \(c \in (a,b)\). Hence, \((Dg)(c) = 0 \implies (Df)(c) = \lambda\). \(\square\)

Definition 2.
\(\mathbb{R}^\# = [-\infty, +\infty]\).

Theorem 3 (L'Hôpital's Rule).
Let \( a, b \in \mathbb{R}^\#, f, g : (a,b) \to \mathbb{R} \) differentiable, and \((Dg)(x) \neq 0\) for all \(x \in (a,b)\). Suppose that \(\frac{(Df)(x)}{(Dg)(x)} \to A \in \mathbb{R}^\# \) as \(x \to a\). If either \(f(x) \to 0\) and \(g(x) \to 0\) as \(x \to a\) or if \(g(x) \to \pm \infty\) as \(x \to a\), then \(\frac{f(x)}{g(x)} \to A\) as \(x \to a\). The analogous statement is true if \(x \to b\).

Proof:
\(Dg \neq 0\) on \((a,b)\) and Darboux's Theorem 1 implies either \(Dg > 0\) or \(Dg < 0\) on \((a,b)\) \(\implies g\) is strictly monotone on \((a,b)\). In particular, if \(g(u) \to 0\) as \(u \to a\), then \(g(u) \neq 0\) for all \(u \in (a,b)\).

Suppose that \(-\infty \leq A < +\infty\). Choose \(q \in \mathbb{R}\) with \(A < q\). There exists \(c \in (a,b)\) such that \(x \in (a,c) \implies \frac{(Df)(x)}{(Dg)(x)} < q\). If \(a < x < y < c\), then Cauchy's Mean Value Theorem shows that there is \(t \in (x,y)\) such that \(\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{(Df)(t)}{(Dg)(t)} < q\). Suppose that \(f(u) \to 0\) and \(g(u) \to 0\) as \(u \to a\). Letting \(x \to a\) we have that \(\frac{f(y)}{g(y)} \leq q\) for any \(y \in (a,c)\). Next, suppose that \(g(u) \to +\infty\) as \(u \to a\). Keeping \(y\) fixed, we can choose a point \(c_1 \in (a,y)\) such that \(g(x) > g(y)\) and \(g(x) > 0\) for \(x \in (a,c_1)\). Now multiplying by \(\frac{g(x)-g(y)}{g(x)}\), we obtain \(\frac{f(x)}{g(x)} < q - q\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}\) for all \(a < x < c_1 < y\). Let \(x \to a\) we have that there is some \(c_2 \in (a,c_1)\) such that \(\frac{f(x)}{g(x)} < q\) for \(a < x < c_2\). A similar argument applies in case \(g(u) \to -\infty\) as \(u \to a\). In every case, for any \(A < q\) there is some \(a < c_2 < b\) such that \(a < x < c_2 \implies \frac{f(x)}{g(x)} \leq q \implies \limsup_{x \to a} \frac{f(x)}{g(x)} \leq q \implies \liminf_{x \to a} \frac{f(x)}{g(x)} \leq A\). A very similar argument also shows that \(-\infty < A \leq +\infty \implies A \leq \liminf_{x \to a} \frac{f(x)}{g(x)}\). Of course a virtually identical proof shows that the analogous statements are true if \(x \to b\). \(\square\)
Warning. Apply L’Hôpital’s Rule, like you would any theorem, only to situations where the hypotheses hold.
Section 6.1. Inverse Functions.

One-to-one functions, examples, horizontal line test, inverses, ”a function has an inverse if and only if it is one-to-one”, finding inverses, examples, derivatives of inverse functions, the derivative rule for inverses, examples.

Derivatives of inverse functions. Use smooth tangent line arguments to show that derivatives of $f^{-1}$ exit. Then $(f^{-1}(f(x)) = x \implies (f^{-1})'(f(x))f'(x) = 1 \implies (f^{-1})'(f(x)) = 1/f'(x) = 1/f'(f^{-1}(f(x))) \implies (f^{-1})'(y) = 1/f'(f^{-1}(y))$ whenever the denominator of the last term is nonzero. Alternatively, and perhaps simpler, $f(f^{-1}(y)) = y \implies f'(f^{-1}(y))(f^{-1})'(y) = 1 \implies (f^{-1})'(y) = 1/f'(f^{-1}(y))$ whenever the denominator is nonzero.

**Example 4**

$f(x) = x^2$ for $x > 0$ and $f^{-1}(y) = \sqrt{y}$ for $y > 0$. $f'(x) = 2x$ and therefore $(f^{-1})'(y) = 1/f'(f^{-1}(y)) = 1/f'(\sqrt{y}) = 1/2\sqrt{y}$.

**Example 5**

$f(x) = x^3 - 2 \implies f'(x) = 3x^2$. Find $(f^{-1})'(6)$. Note that $f(2) = 6 \implies f^{-1}(6) = 2$. $(f^{-1})'(6) = 1/f'(f^{-1}(6)) = 1/f'(2) = 1/12$.

**Exercise 1**

Look over Section 6.1 and the associated exercises. You should be able to do #17 - #28, #33 - #46, #49, #50, pp. 406 - 407.

Graded:

#22, #27, #41, #49, #50, pp. 406 - 407. No graphing.

These homeworks are due at the beginning of class on Wednesday, January 23, 2019. Remember that homeworks make up circa 40% of your grade.
Section 6.2* & 6.3*. The Natural Logarithmic and National Exponential Functions.

Logarithms and their practical importance.

Definition 6.
If $x > 0$, define $\log(x) \equiv \log_e(x) \equiv \ln(x) = \int_1^x dw/w$.

Theorem 7.
$\ln(1) = 0$, $x > 1 \implies \ln(x) > 0$, $0 < x < 1 \implies \ln(x) < 0$, $\ln(1/x) = -\ln(x)$, $\ln'(x) = 1/x > 0$ so $\ln(x)$ is strictly increasing and $\ln''(x) = -1/x^2 < 0$, so the graph of $\ln(x)$ is concave downward.

Proof:
Use the Fundamental Theorem of Calculus and a change of variables. □

Example 8.
$h(x) = \ln(2x) \implies h'(x) = 1/x$.

Theorem 9.
If $a > 0$ and $b > 0$, then:
(1) $\ln(ab) = \ln(a) + \ln(b)$;
(2) $\ln(1/b) = -\ln(b)$;
(3) $\ln(a/b) = \ln(a) - \ln(b)$;
(4) $\ln(a^n) = n \ln(a)$ for every integer $n$;
(5) $\ln(a^{1/q}) = (1/q) \ln(a)$ for all $q > 0$;
(6) $\ln(a^r) = r \ln(a)$ for all $r \in \mathbb{Q}$.

Proof:
(1) Let $h(x) = \ln(ax) \implies h'(x) = (\ln'(ax))(ax)' = \frac{1}{ax} \cdot a = 1/x = \ln'(x) \implies \ln(ax) = h(x) = \ln(x) + C \implies \ln(a) = \ln(a \cdot 1) = \ln(1) + C = 0 + C = C \implies \ln(ax) = \ln(x) + \ln(a) \implies \ln(ab) = \ln(a) + \ln(b)$.
(2) $0 = \ln(1) = \ln((1/a)a) = \ln(1/a) + \ln(a) \implies \ln(1/a) = -\ln(a)$.
(3) $\ln(a/b) = \ln(a(1/b)) = \ln(a) + \ln(1/b) = \ln(a) - \ln(b)$.
(4) $\ln(a^1) = \ln(a) = 1 \cdot \ln(a)$ and inductively $\ln(a^{n+1}) = \ln(a^n \cdot a) = \ln(a^n) + \ln(a) = n \ln(a) + \ln(a) = (n + 1) \ln(a)$ for every $n \geq 1$. $\ln(a^0) = \ln(1) = 0 = 0 \cdot \ln(a)$ and $\ln(a^{-n}) = \ln(1/a^n) = -n \ln(a) = -n(\ln(a)) = (-n) \ln(a)$.
(5) $\ln(a) = \ln((a^{1/q})^q) = q \ln(a^{1/q}) \implies \ln(a^{1/q}) = (1/q) \ln(a)$.
(6) If $r = p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then $\ln(a^r) = \ln((a^{1/q})^p) = p \ln(a^{1/q}) = p(1/q) \ln(a) = (p/q) \ln(a) = r \ln(a)$. □
Lemma 10.
\[ \ln(x) \to +\infty \text{ as } x \to +\infty \text{ and } \ln(x) \to -\infty \text{ as } x \downarrow 0+. \]

Proof:
\[ \ln(2) = \int_1^2 \frac{dw}{w} > 0 \implies \ln(2^n) = n \ln(2) \to +\infty \text{ as } n \to +\infty. \] Since \( \ln(x) \uparrow \), we have that \( \ln(x) \to +\infty \text{ as } x \to +\infty \). \( \ln(x) \to -\infty \text{ as } x \downarrow 0+ \) since \( \ln(x) = -\ln(1/x) \). \( \square \)

Theorem 11.
\( \ln : \langle 0, +\infty \rangle \to \mathbb{R} \) is one-to-one and onto and therefore \( \ln^{-1} \) exists.

Proof:
\( \ln \uparrow \) so \( \ln \) is one-to-one. Since \( \ln \) takes arbitrarily large positive and negative values and is continuous since it is differentiable, every horizontal line cuts the graph of \( \ln \). Therefore \( \ln : \langle 0, +\infty \rangle \to \mathbb{R} \) is onto. \( \square \)

Theorem 12.
\[ \int f'/f = \ln(|f|) + C \text{ and } \int (1/u)du = \ln(|u|) + C \]

Proof:
Use the chain rule on \( \ln(f) \) if \( f > 0 \) and on \( \ln(-f) \) if \( f < 0 \). \( \square \)

Example 13.
\[ \int_0^1 \frac{2x}{x^2-5} dx = \ln(4/5). \]

Example 14.
\[ \int_{-\pi/2}^{\pi/2} \frac{4\cos(\theta)}{3+2\sin(\theta)} d\theta = 2 \ln(5). \]

Example 15.
\[ \int \tan(x)dx = \ln(|\sec(x)|) + C \text{ and } \int \cot(x)dx = -\ln(|\csc(x)|) + C. \]

Example 16.
\[ \int \sec(x)dx = \ln(|\sec(x) + \tan(x)|) + C \text{ and } \int \csc(x)dx = -\ln(|\csc(x) + \cot(x)|) + C. \]

Proof:
For \( \int \sec(x)dx \) multiply top and bottom by \( \sec(x) + \tan(x) \) and use \( \int f'/f = \ln(|f|) + C. \) Similarly for \( \int \csc(x)dx \). \( \square \)

Example 17.
\[ \int_0^{\pi/6} \tan(2x)dx = \ln(2)/2. \]
Example 18 (Logarithmic Differentiation).
Find \( \frac{dy}{dx} \) if \( y = \frac{(x^2+1)(x+3)^{1/2}}{x-1} \) for \( x > 1 \).

Example 19.
\[
\int \ln(x)\,dx = x \ln(x) - x + C
\]

Example 20.
Let \( f(x) = (\ln(x^3 + 6x^2 + 4))^{17} \). Compute \( f'(x) \).

Recall that \( \ln : \langle 0, +\infty \rangle \to \mathbb{R} \) is one-to-one and onto. Hence, \( \ln^{-1} : \mathbb{R} \to \langle 0, +\infty \rangle \) exists.

Definition 21.
\[ \exp(x) = \ln^{-1}(x). \]

Theorem 22.
If \( x, y \in \mathbb{R} \), then:

1. \( \exp(0) = 1 \);
2. \( \exp(x + y) = \exp(x)\exp(y) \);
3. \( \exp(-x) = 1/\exp(x) \);
4. \( \exp(rx) = (\exp(x))^r \) for all \( r \in \mathbb{Q} \);
5. \( (\exp)'(x) = \exp(x) \);
6. \( a' = \exp(r \ln(a)) \) for all \( a > 0 \) and \( r \in \mathbb{Q} \).

Proof:
1. \( \ln(\exp(0)) = 0 = \ln(1) \implies \exp(0) = 1 \).
2. \( \ln(\exp(x + y)) = x + y \) and \( \ln(\exp(x)\exp(y)) = \ln(\exp(x)) + \ln(\exp(y)) = x + y \implies \exp(x + y) = \exp(x)\exp(y) \).
3. \( 1 = \exp(0) = \exp(x + (-x)) = \exp(x)\exp(-x) \implies \exp(-x) = 1/\exp(x) \).
4. \( \ln(\exp(rx)) = rx \) and \( \ln((\exp(x))^r) = r \ln(\exp(x)) = rx \implies \exp(rx) = (\exp(x))^r \).
5. \( (\exp)'(x) = 1/(\ln)'(\exp(x)) = 1/(1/\exp(x)) = \exp(x) \).
6. \( \ln(a') = r \ln(a) \) and \( \ln(\exp(r \ln(a))) = r \ln(a) \implies a' = \exp(r \ln(a)). \)

Corollary 23.
If \( f \) is a real-valued function, then \( (\exp(f))' = \exp(f)f' \) and \( \int \exp(f)\,f' = \exp(f) + C \). In particular, if \( a \in \mathbb{R} \), \( a \neq 0 \), then \( (\exp(ax))' = a \exp(ax) \) and \( \int \exp(ax)\,dx = \exp(ax)/a + C \).

Proof:
Use chain rule. \( \square \)
Exercise 2.
Look over Section 6.2*. You should be able to do
#1 - #10, #15 - #44, #47 - #52, #65 - #74, #77 - #81, pp. 445 - 447.
Graded:
#9, #27, #31, #44, #50, #64, #70, pp. 445 - 447.
These homeworks are due at the beginning of class on Wednesday, January 30, 2019. Remember that homeworks make up circa 40% of your grade.

Exercise 3.
Look over Section 6.3*. You should be able to do
#2 - #12, #15 - #16, #21 - #63, #67 - #70, #83 - #94, pp. 452 - 455.
Graded:
#26, #27, #41, #48, #59, #69, #93, pp. 452 - 455.
These homeworks are due at the beginning of class on Wednesday, January 30, 2019. Remember that homeworks make up circa 40% of your grade.
Section 6.4*. General Logarithmic and Exponential Functions.

**Definition 24.**
\[ e = \exp(1) > 0 \iff e = \ln^{-1}(1) \iff \ln(e) = 1. \]

**Lemma 25.**
If \( r \in \mathbb{Q} \), then \( e^r = \exp(r) = \ln^{-1}(r) \).

Proof:
\[ e^r = (\exp(1))^r = \exp(r \cdot 1) = \exp(r). \]

Heuristics. If \( a > 0 \) and \( b \in \mathbb{R} \), then \( y = a^b \iff \ln(y) = \ln(a^b) = b \ln(a) \iff a^b = y = \exp(\ln(y)) = \exp(b \ln(a)) \).

**Definition 26.**
If \( a > 0 \) and \( b \in \mathbb{R} \), then \( a^b = \exp(b \ln(a)) \).

**Proposition 27.**
\( \ln(a^b) = b \ln(a) \).

Proof:
\[ \ln(a^b) = \ln(\exp(b \ln(a))) = b \ln(a). \]

**Proposition 28.**
\( e^x = \exp(x) \).

Proof:
\[ e^x = \exp(x \ln(e)) = \exp(x \cdot 1) = \exp(x). \]

**Examples 29.**
Compute \( \frac{d}{dx} e^{-x} \) and \( \frac{d}{dx} e^{\sin(x)} \).

Heuristics. If \( a > 0 \) and \( b > 0 \), then we want \( a^{\log_a(b)} = b \iff \ln(a^{\log_a(b)}) = \ln(b) \iff \log_a(b) \ln(a) = \ln(b) \iff \log_a(b) = \ln(b)/\ln(a) \).

**Definition 30.**
If \( a > 0 \), \( a \neq 1 \), and \( b > 0 \), let \( \log_a(b) = \ln(b)/\ln(a) \).
**Theorem 31.**

Let $a > 0$, $a \neq 1$, $b \in \mathbb{R}$, $x > 0$, and $y > 0$. Then:

1. $\log_a(1) = 0$;
2. $\log_a(a) = 1$;
3. $\log_e(x) = \ln(x)$;
4. $\log_a(xy) = \log_a(x) + \log_a(y)$;
5. $\log_a(1/x) = -\log_a(x)$;
6. $\log_a(x/y) = \log_a(x) - \log_a(y)$;
7. $\log_a(x^b) = b \log_a(x) \implies \log_a(a^b) = b$;
8. $(\log_a)'(x) = 1/x \ln(a)$;
9. $(d/dx)(\log_x(a)) = -\ln(a)/x \ln^2(x)$